Portfolio Optimization

BUSI 721: Data-Driven Finance I

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Outline

1. Review of portfolio expected returns and risks
2. Define mean-variance efficient and global minimum variance portfolios
3. Example of quadratic programming with cvxopt
4. SPY, IEF, and GLD returns
5. GMV portfolio of SPY, IEF, and GLD
6. Mean-variance efficient portfolios of SPY, IEF, and GLD
7. Include cash with SPY, IEF, and GLD
8. Sharpe ratios and the tangency portfolio

Review: Portfolio expected return

- With n risky assets,

$$\sum_{i=1}^{n} w_i \mu_i + \left(1 - \sum_{i=1}^{n} w_i\right) r_f$$

- where $r_f =$ money market rate if $\sum w_i < 1$ and
- $r_f =$ margin loan rate if $\sum w_i > 1$ and
- we are ignoring interest drag and short borrowing fee if any of the w_i are negative.

Review: Reg T

- Initial margin requirement: when positions are put on,

$$\sum |w_i| \leq 2$$

- Afterwards, brokers impose maintenance margin requirements.
- Example: invest 1,000, borrow 1,000, buy 20 shares of $\$$ 100 stock
  - $\sum w_i = 2$
  - Stock price falls to 75.
  - Now have 1,500 of stock.
  - Portfolio value is 1,500 - 1,000 = 500. Weight on stock is 1,500 / 500 = 3.
  - Maybe get margin call.

Review: Portfolio variance

- Two assets:

$$w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho\sigma_1\sigma_2$$

- Three assets:

$$w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2$$

$$+2w_1w_2\rho_{12}\sigma_1\sigma_2 + 2w_1w_3\rho_{13}\sigma_1\sigma_3 + 2w_2w_3\rho_{23}\sigma_2\sigma_3$$

- Any number of assets:

$$w'\Sigma w$$

Matrix multiplication

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

$$\begin{pmatrix} g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ga + hc & gb + hd \end{pmatrix}$$

$$\begin{pmatrix} w_1 & w_2 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 \end{pmatrix} \begin{pmatrix} \sigma_1^2 w_1 + \rho\sigma_1\sigma_2 w_2 \\ \rho\sigma_1\sigma_2 w_1 + \sigma_2 w_2 \end{pmatrix}$$

$$= \sigma_1^2 w_1^2 + \rho\sigma_2\sigma_2 w_1 w_2 + \rho\sigma_2\sigma_2 w_1 w_2 + \sigma_2 w_2^2$$

2. Mean-Variance Frontier and GMV Portfolio

Mean-Variance Frontier

- Mean-variance frontier is the set of portfolios that have the least risk among all portfolios that have their expected return
- Minimum risk problem: minimize variance subject to constraints:
    - achieve a target expected return
    - $\sum w_i = 1$
    - possibly $w_i \geq 0$ or Reg T
- We can vary the target expected return and trace out the mean-variance frontier
- Some points on the frontier may be inefficient (meaning you can do better on both risk and expected return) because the target expected return is too low.

Global minimum variance portfolio

- Solve the minimization problem without a target expected return
- This portfolio (GMV portfolio) has the least risk among all portfolios
- Frontier portfolios are efficient (meaning you can't do better on both risk and expected return) if and only if the target expected return $\geq$ expected return of GMV portfolio.

3. Quadratic programming

- Finding efficient portfolios and finding the GMV portfolio are examples of quadratic programming
- Minimize or maximize a quadratic function (squares and products and linear terms)
    - Subject to linear inequality constraints
    - And subject to linear equality constraints

Quadratic Programming Example

minimize

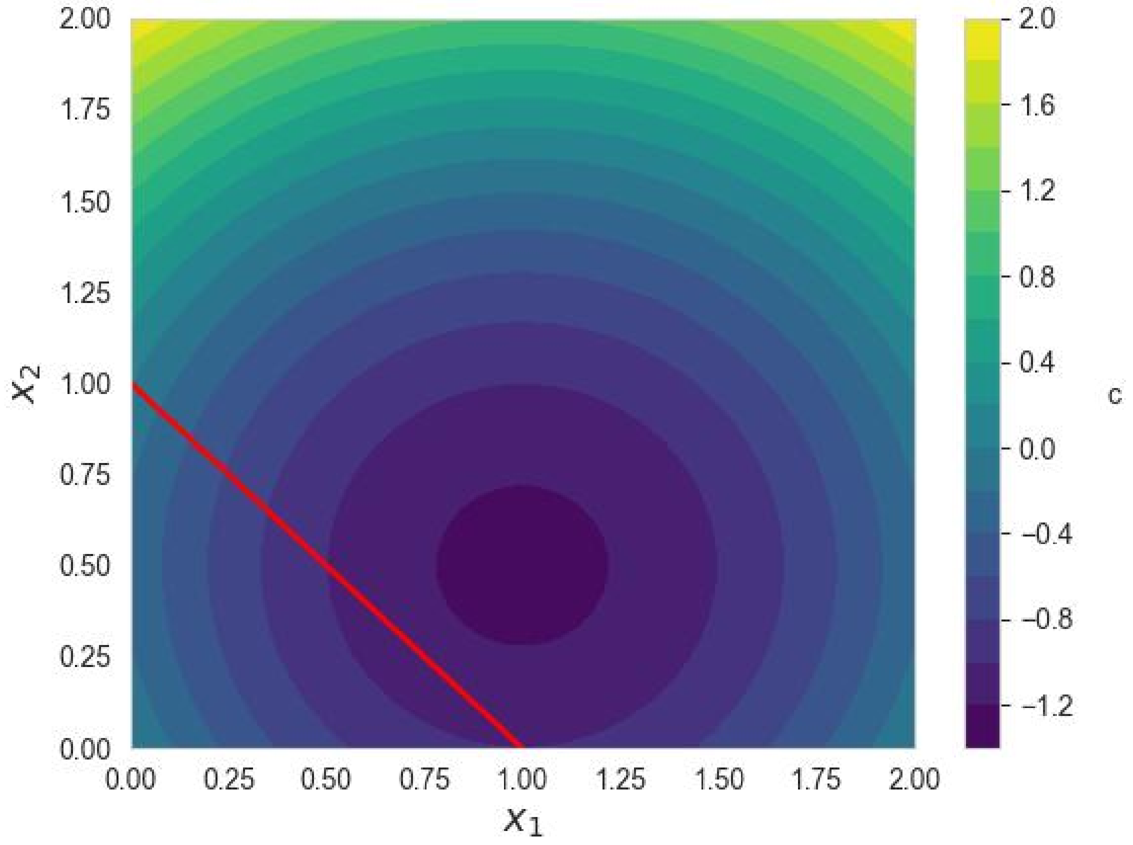
$$x_1^2 + x_2^2 - 2x_1 - x_2$$

subject to

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$x_1 + x_2 = 1$$



2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
x_2
0.00 0.25 0.50 0.75 1.00 1.25 1.50 1.75 2.00
x_1
2.0
1.6
1.2
0.8
0.4
0.0
−0.4
−0.8
−1.2
c

Notation of cvxopt

minimize

$$\frac{1}{2}x'Px + q'x$$

subject to

$$Gx \leq h$$

and

$$Ax = b$$

Our example

$$P = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \Rightarrow \quad \frac{1}{2} x' P x = x_1^2 + x_2^2$$

$$q = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \quad \Rightarrow \quad q' x = -2x_1 - x_2$$

$$G = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Rightarrow \quad Gx = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}$$

$$h = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \end{pmatrix} \quad \Rightarrow \quad Ax = x_1 + x_2$$

$$b = \begin{pmatrix} 1 \end{pmatrix}$$

Define arrays

In [3]:

```python
P = np.array(
    [
        [2., 0.],
        [0., 2.]
    ]
)
q = np.array([-2., -1.]).reshape(2, 1)
G = np.array(
    [
        [-1., 0.],
        [0., -1.]
    ]
)
h = np.array([0., 0.]).reshape(2, 1)
A = np.array([1., 1.]).reshape(1, 2)
b = np.array([1.]).reshape(1, 1)
```

Solve

In [4]:

```python
from cvxopt import matrix
from cvxopt.solvers import qp

sol = qp(
    P=matrix(P),
    q=matrix(q),
    G=matrix(G),
    h=matrix(h),
    A=matrix(A),
    b=matrix(b)
)
np.array(sol["x"])
```

```
     pcost       dcost       gap    pres   dres
 0: -1.1111e+00 -2.2222e+00  1e+00  1e-16  1e+00
 1: -1.1231e+00 -1.1680e+00  4e-02  1e-16  4e-02
 2: -1.1250e+00 -1.1261e+00  1e-03  2e-16  3e-04
 3: -1.1250e+00 -1.1250e+00  1e-05  6e-17  3e-06
 4: -1.1250e+00 -1.1250e+00  1e-07  3e-16  3e-08
Optimal solution found.
```

Out[4]:

```
array([[0.7499999],
       [0.2500001]])
```

4. Stock, Bond, and Gold ETFs

- SPY, IEF, and GLD adjusted closing prices from Yahoo
- Downsample to monthly
- Percent changes are monthly returns
- Compute historical means and covariance matrix

In [5]:

```
import yfinance as yf

tickers = ["SPY", "IEF", "GLD"]
prices = yf.download(tickers, start="1970-01-01")["Adj Close"]
prices = prices.resample("M").last()
rets = prices.pct_change().dropna()
rets.head(3)
```

```
[*********************100%%**********************]  3 of 3 completed
```

Out[5]:

	GLD	IEF	SPY
Date			
2004-12-31	-0.029255	0.011674	0.030121
2005-01-31	-0.036073	0.008710	-0.022421
2005-02-28	0.031028	-0.013683	0.020904

Means, risks and correlations

In [6]:
```python
12 * rets.mean()
```

Out[6]:
```
GLD      0.087096
IEF      0.031683
SPY      0.100341
dtype: float64
```

In [7]:
```python
np.sqrt(12) * rets.std()
```

Out[7]:
```
GLD     0.169435
IEF     0.064872
SPY     0.150749
dtype: float64
```

In [8]:
```
rets.corr()
```

Out[8]:

	GLD	IEF	SPY
GLD	1.000000	0.317975	0.084318
IEF	0.317975	1.000000	-0.121379
SPY	0.084318	-0.121379	1.000000

In [9]:

```
mu = rets.mean().to_numpy()
Sigma = rets.cov().to_numpy()
```

5. GMV Portfolio of Stocks, Bonds, and Gold

GMV minimization problem

minimize

$$\frac{1}{2}w'\Sigma w$$

subject to

$$\sum w_i = 1 \quad \Leftrightarrow \quad \iota' w = 1$$

where ι is a column vector of ones.

cvxopt formulation

- $P = \Sigma$
- $q = 0$

$$A = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

$$b = \begin{pmatrix} 1 \end{pmatrix}$$

Define arrays

In [10]:

```
P = Sigma
q = np.zeros((3, 1))
A = np.ones((1, 3))
b = np.ones((1, 1))
```

Compute the GMV portfolio

In [11]:

```python
sol = qp(
    P=matrix(P),
    q=matrix(q),
    A=matrix(A),
    b=matrix(b)
)

import pandas as pd
gmv = pd.Series(sol["x"], index=rets.columns)
gmv
```

Out[11]:

```
GLD   -0.001301
IEF    0.817025
SPY    0.184276
dtype: float64
```

Risk and expected return of GMV portfolio

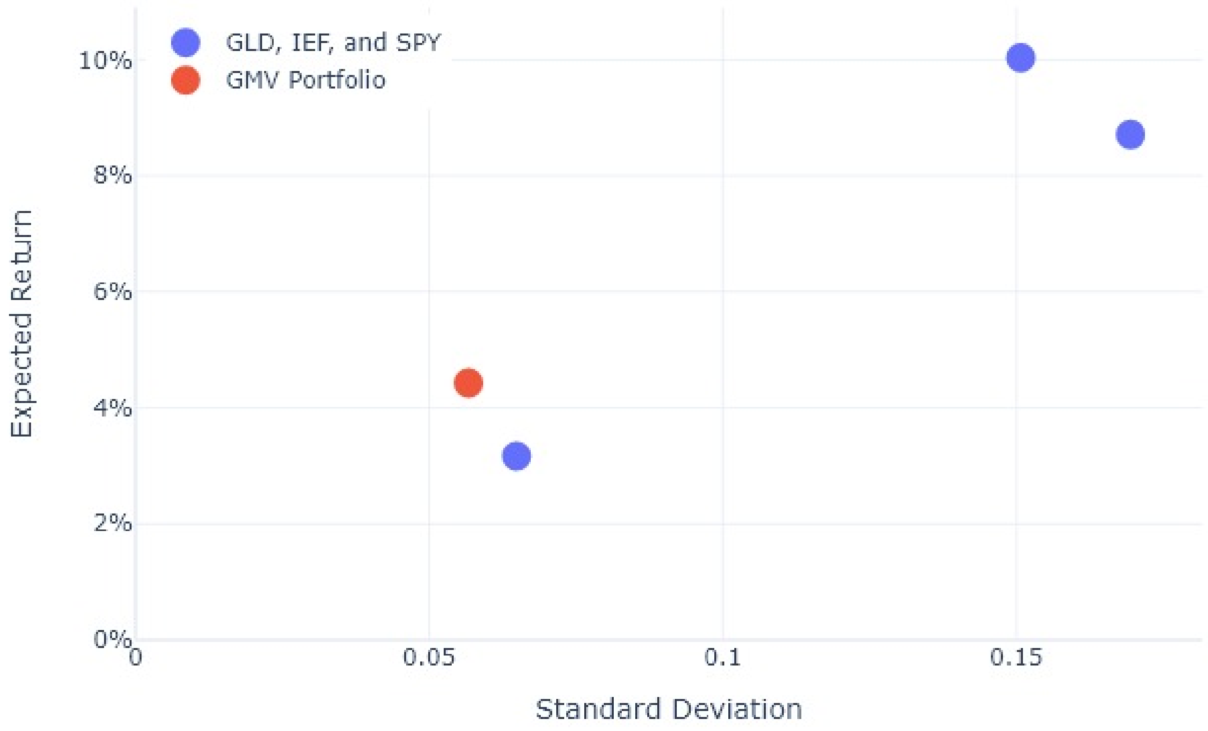
In [12]:

```
w = gmv.to_numpy()
print(f"\nGMV annualized std dev is {np.sqrt(12*w@Sigma@w):.2%}")
print(f"GMV annualized mean is {12*mu@w: .2%}")

print(f"\nIEF annualized std dev is {np.sqrt(12)*rets.IEF.std():.2%}")
print(f"IEF annualized mean is {12*rets.IEF.mean():.2%}")
```

```
GMV annualized std dev is 5.67%
GMV annualized mean is  4.43%

IEF annualized std dev is 6.49%
IEF annualized mean is 3.17%
```


GLD, IEF, and SPY
GMV Portfolio
Expected Return
10%
8%
6%
4%
2%
0%
0
0.05
0.1
0.15
Standard Deviation

6. Efficient portfolios of SPY, IEF, and GLD

Minimize risk with target expected return

minimize

$$\frac{1}{2}w'\Sigma w$$

subject to

$$\mu' w = r$$

$$\iota' w = 1$$

where $r =$ target expected return and ι is a column vector of ones.

cvxopt formulation

- $P = \Sigma$
- $q = 0$

$$A = \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ 1 & 1 & 1 \end{pmatrix}$$

$$b = \begin{pmatrix} r \\ 1 \end{pmatrix}$$

Define arrays

In [14]:

```
# example target monthly expected return
r = 0.06/12

P = Sigma
q = np.zeros((3, 1))
A = np.array(
    [
        mu,
        [1., 1., 1.]
    ]
)
b = np.array([r, 1]).reshape(2, 1)
```

Compute the efficient portfolio

In [15]:

```python
sol = qp(
    P=matrix(P),
    q=matrix(q),
    A=matrix(A),
    b=matrix(b)
)
efficient = pd.Series(sol["x"], index=rets.columns)
efficient
```

Out[15]:

```
GLD     0.115466
IEF     0.565289
SPY     0.319245
dtype: float64
```

GLD, IEF, and SPY
GMV Portfolio
Efficient Portfolio
10%
8%
6%
4%
2%
0%
0
0.05
0.1
0.15
Expected Return
Standard Deviation

7. Include cash with SPY, IEF, and GLD

Target expected return with cash

- Expected return is

$$\mu' w + (1 - \iota' w) r_f = r_f + (\mu - r_f \iota')w$$

- Equals target expected return r if and only if

$$(\mu - r_f \iota)' w = r - r_f$$

- So,

$$A = \begin{pmatrix} \mu_1 - r_f & \mu_2 - r_f & \mu_3 - r_f \end{pmatrix}$$

$$b = \begin{pmatrix} r - r_f \end{pmatrix}$$

Define arrays

In [17]:

```
# example monthly interest rate
rf = 0.03/12

# example target expected return
r = 0.06/12

P = Sigma
q = np.zeros((3, 1))
A = (mu - rf*np.ones(3)).reshape(1, 3)
b = np.array([r-rf]).reshape(1, 1)
```

Compute the efficient portfolio

In [18]:

```python
sol = qp(
    P=matrix(P),
    q=matrix(q),
    A=matrix(A),
    b=matrix(b)
)
efficient_with_cash = pd.Series(sol["x"], index=rets.columns)
efficient_with_cash
```

Out[18]:

```
GLD     0.176189
IEF    -0.027139
SPY     0.284129
dtype: float64
```


GLD, IEF, and SPY
GMV Portfolio
Efficient Portfolio
Efficient Portfolio with Cash
Expected Return
10%
8%
6%
4%
2%
0%
0
0.05
0.1
0.15
Standard Deviation

8. Sharpe Ratios and the Tangency Portfolio

Sharpe ratio

- The Sharpe ratio is defined as

$$\frac{\text{Expected Return} - \text{Risk-Free Rate}}{\text{Standard Deviation}}$$

- To annualize a monthly Sharpe ratio,
  - numerator should be multiplied by 12,
  - denominator should be multiplied by $\sqrt{12}$
  - so ratio should be multiplied by $\sqrt{12}$

Sharpe ratios of SPY, IEF, and GLD

In [20]:

```
sharpes = np.sqrt(12)*(rets.mean() - rf) / rets.std()
sharpe_efficient = np.sqrt(12)*(r - rf) / np.sqrt(w@Sigma@w)

print(f"SPY = {sharpes.SPY:.2%}")
print(f"IEF = {sharpes.IEF:.2%}")
print(f"GLD = {sharpes.GLD:.2%}")
print(f"Efficient portfolio with cash = {sharpe_efficient:.2%}")
```

```
SPY = 46.66%
IEF = 2.59%
GLD = 33.70%
Efficient portfolio with cash = 55.43%
```

Geometry of Sharpe ratios

- Sharpe ratio is slope of line connecting (std dev=0, mean=rf) with the (std dev, mean) of the asset or portfolio
- Efficient portfolios with cash all have the same Sharpe ratio, so they all lie on the same line
- This is the maximum possible Sharpe ratio - the line is the furthest northwest in the (std dev, mean) diagram.

Tangency portfolio

- Tangency portfolio is an efficient portfolio with cash that does not use cash
- It is efficient with or without cash
- It is the point at which the line with maximum Sharpe ratio just touches the frontier without cash
- We should hold the tangency portfolio with or without cash
- Will look at margin loans later

Tangency portfolio of SPY, IEF, and GLD

In [21]:

```python
tang = w / np.sum(w)
pd.Series(tang, index=rets.columns).round(3)
```

Out[21]:

```
GLD    0.407
IEF   -0.063
SPY    0.656
dtype: float64
```